Improved upper bound on the number of minimal dominating sets in pendant graphs

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Abstract

Given a simple graph on n vertices, currently 1.7159^n is the best upper bound on the number of minimal dominating sets. This bound has been improved for some classes of graphs. In this article, the bound 1.7159^n is improved for the class of simple loop-free connected graphs having pendant vertices, leading up to the corresponding results for simple loop-free connected hypergraphs.

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1. Introduction

It was shown in [5] that the number of minimal dominating sets in a given graph on n vertices is at most 1.7159^n. Subsequently, this bound was improved in some special classes of graphs. These are 1.6181^n for chordal graphs; 1.4656^n for split graphs and for proper interval graphs; 1.4423^n for trivially perfect graphs, all in [3]. Moreover, 1.4656^n in the case of trees [6]. Note that all these expressions are in terms of the number of vertices (n).

A motivating question for this research work was: Within a class of graphs, can different graphs vary in their upper bounds on the number of their minimal dominating sets? If so, can it be realized through an upper bound expression that depends on factors besides the number of vertices?

The contribution of this article is an improved upper bound in the form δ (1.7159^n) (with $0 < \delta < 1$) that answers the preceding question in the affirmative, in the class of simple loop-free connected [1] graphs having at least one pendant vertex. This class includes the class of all trees. The number δ depends on the number of pendant vertices and also on how these are placed in the graph. In any case, $\delta < 0.9225$. Much of the motivation for this research work comes from [3], [4] and [5].

The cardinality [7] of a finite set V is denoted by $|V|$. A *simple hypergraph* [2] is an ordered couple H = (V, E) where: (i) V is a nonempty finite set and (ii) E is a set of nonempty subsets of V such that $\cup_{X \in E} X = V$. Each member of V is a *vertex*; and each member of E is a *hyperedge* (or, an *edge*). A hyperedge X with $|X| = 1$ is a *loop*. A hypergraph is *loop-free* if $|X| > 1$ for each hyperedge X.

A *simple loop-free graph* is a simple loop-free hypergraph G = (V, E) with the additional stipulation that $|X| = 2$ for each hyperedge X. If {x, y} is an edge in G, then x and y are the *end points* of this edge. If x, y ∈ V are distinct, then x is *adjacent* to y in G if {x, y} ∈ E.

If D ⊂ V then D is a *dominating set* (in G) if, each x ∈ V is either in D or is adjacent to some y ∈ D. A dominating set D is a *minimal dominating set* if no proper subset of D is a dominating set.

2. Improving the upper bound – for pendant graphs

If y is a vertex in G, then the set $N(y) = \{x \in V - \{y\} \mid x \text{ is adjacent to } y\}$ is the *neighborhood* of y in G, and a *neighbor* of y is a member of N(y). If A ⊂ V then the set N(A) is the neighborhood of A in G, and N(A) = ∪ N(x) as x runs over A. The integer $|N(y)|$ is the *degree* of y in G, and is denoted by dy (or, dy(G)). If dy = 1 then y is a *pendant* vertex in G. G is a *pendant graph* if G has a pendant vertex. The *pendant count* of a vertex y (in G) is denoted by π (y) and is the number of pendant vertices that are adjacent to y.

2.1: Theorem [5]: Every graph on n vertices contains at most 1.7159^n minimal dominating sets.

2.2: Proposition: Let G = (V, E) be a simple, loop-free, connected pendant graph; b be a vertex in G with $\pi(b) \geq 1$; and D be a minimal dominating set in G. Then:
(i) If $b \in D$, then D does not contain any pendant neighbor of b; and
(ii) if $b \notin D$, then D contains all the pendant neighbors of b.
Proof. Let $P(b) = \{a_1, \ldots, a_k\}$be the set of all the pendant neighbors of b.
(i) Let $b \in D$. If any $a_j \in P(b) \cap D$, then $D - \{a_j\}$ would be a dominating set. Contradiction.
(ii) Assume $b \notin D$. Were some $a_j \notin D$ then a_j would not be adjacent to any vertex in D. Contradiction. ■

2.3: Proposition: Let G be as in 2.2 and let:
(i) $S = \{z \in G \mid \pi(z) \geq 1\} = \{z_1, \ldots, z_s\}$, say;
(ii) $P(z_j) = \{y \in N(z_j) \mid dy = 1\}$, for j = 1 through s;
(iii) $P = \cup P(z_j)$ (j = 1 through s);
(iv) $W_1 = \{y \in V \mid y \in N(S) - P\}$;
(v) $W_2 = \{y \in V \mid y \notin N(S)\}$;
(vi) $W = W_1 \cup W_2$; and
(vii) DOM(G) be the number of minimal dominating sets in G.
Suppose $W = \varphi$. Then $DOM(G) < 1.4143^n$, where $n = |V|$.
Proof. Let $\mathcal{D}(G)$ be the set of all the minimal dominating sets in G; $\mathcal{D}_1(G) = \{D \in \mathcal{D}(G) \mid D \cap S = \varphi\}$ and $\mathcal{D}_2(G) = \{D \in \mathcal{D}(G) \mid D \cap S \neq \varphi\}$. Then $\mathcal{D}(G) = \mathcal{D}_1(G) \cup \mathcal{D}_2(G)$ is a partition [7] of $\mathcal{D}(G)$, and so $DOM(G) = |\mathcal{D}_1(G)| + |\mathcal{D}_2(G)|$. Also, $V = S \cup P(z_1) \cup \ldots \cup P(z_s)$ is a partition of V since $W = \varphi$.
(i) Let $D \in \mathcal{D}_1(G)$ be given. By 2.2, $P \subset D$, from which at once P = D (owing to $W = \varphi$), leading straight to $\mathcal{D}_1(G) = \{P\}$; that is, $|\mathcal{D}_1(G)| = 1$.
(ii) Let $D \in \mathcal{D}_2(G)$ be given. Then $P(z_j) \subset D$ for each j such that $z_j \notin D$; and $P(z_i) \cap D = \varphi$ for each i such that $z_i \in D$. Let T be a given nonempty subset of S; and $D_1, D_2 \in \mathcal{D}_2(G)$ such that $D_1 \cap S = D_2 \cap S = T$. Then $D_1 = D_2$. Consequently, for each $k \in \{1, \ldots, s\}$, there are precisely ${}_sC_k$ minimal dominating sets D such that $D \in \mathcal{D}_2(G)$ and $|D \cap S| = k$. Then $|\mathcal{D}_2(G)| = {}_sC_1 + \ldots + {}_sC_s = 2^s - 1$. Since $|P(z_j)| \geq 1$ for each j = 1 through s, and $n = s + |P(z_1)| + \ldots + |P(z_s)|$, it follows that $2s \leq n$, giving $|\mathcal{D}_2(G)| \leq (\sqrt{2})^n - 1$. And so $DOM(G) = |\mathcal{D}_1(G)| + |\mathcal{D}_2(G)| \leq (\sqrt{2})^n < 1.4143^n$. ■

2.4: Proposition: Let G, S, $P(z_j)$, P, W_1, W_2, W and DOM(G) all be as in 2.3. Assume $W \neq \varphi$. Let $n = |V|$, $s = |S|$ and $p = |P|$. Then:
(i) $2.7159^s < 1.7159^{s+p}$; and
(ii) $DOM(G) \leq (2.7159^s)(1.7159^{n-s-p})$.
Proof. (i) Clearly $p \geq s$ since $p = |P| = |P(z_1)| + \ldots + |P(z_s)|$. Hence $s + p \geq 2s$. Consequently $(1 + \alpha)^s / \alpha^{s+p} \leq [(1 + \alpha) / \alpha^2]^s$, where $\alpha = 1.7159$. And $(1 + \alpha) / \alpha^2 < 0.9225 < 1$, from which (i) follows.

(ii) Let [W] denote the subgraph of G induced [1] by W. Let $\mathcal{D}_1(G)$ and $\mathcal{D}_2(G)$ be as in the proof of 2.3. Let $D \in \mathcal{D}_1(G)$ be given. Then $P \subset D$ and $P \neq D$, and so let $D - P = X$. Then $X \subset W$, and X is a dominating set in [W]. Were $X - \{y\}$ a dominating set in [W] for some $y \in X$, then $(X - \{y\}) \cup P$ would be a dominating set in G – impossible since (X – {y}) ∪ P is a proper subset of D. So X is a minimal dominating set in [W].

If $D_1, D_2 \in \mathcal{D}_1(G)$ are distinct, then $P \subset D_1 \cap D_2$; also $D_1 - P$ and $D_2 - P$ are distinct minimal dominating sets in [W]. Consequently, $|\mathcal{D}_1(G)| \leq$ DOM([W]). Invoking 2.1 now gives DOM([W]) $\leq 1.7159^{n-s-p}$ since $|W| = n - s - p$; so $|\mathcal{D}_1(G)| \leq 1.7159^{n-s-p}$.

Next, for k ∈ {1, . . ., s} let $\mathcal{D}_2(G)_k = \{D \in \mathcal{D}_2(G) \mid |D \cap S| = k\}$. Note that $\mathcal{D}_2(G) = \mathcal{D}_2(G)_1 \cup \ldots \cup \mathcal{D}_2(G)_s$ is a partition of $\mathcal{D}_2(G)$. Given $D \in \mathcal{D}_2(G)_k$, write X = D – P and M = W ∪ (D ∩ S). Then X is a dominating set in the subgraph [M] of G. Were X – {y} a dominating set in [M] for some y ∈ X – S, then (X – {y}) ∪ (D ∩ P) would be a dominating set in G, impossible because (X – {y}) ∪ (D ∩ P) is a proper subset of D. Then either X is a minimal dominating set in [M] or X – B is a minimal dominating set in [M] for some B ⊂ D ∩ S. Invoking 2.1, it turns out that $|\mathcal{D}_2(G)_k| \leq (1.7159^{|W| + k})\ {}_sC_k$, since $|D \cap S| = k$ is possible in ${}_sC_k$ distinct ways. Since $|\mathcal{D}_2(G)| = \sum_{k=1 \text{ to } s} |\mathcal{D}_2(G)_k|$, it comes to $|\mathcal{D}_2(G)| \leq \alpha^{|W|}({}_sC_1\,\alpha + \ldots + {}_sC_s\,\alpha^s)$, with α = 1.7159. So $|\mathcal{D}_2(G)| \leq \alpha^{n-s-p}[(1+\alpha)^s - 1]$ since $|W| = n - s - p$.

Thus $|\mathcal{D}(G)| = |\mathcal{D}_1(G)| + |\mathcal{D}_2(G)| \leq \alpha^{n-s-p} + \alpha^{n-s-p}[(1+\alpha)^s - 1]$; that is, DOM(G) $\leq (2.7159^s)(1.7159^{n-s-p})$, completing the proof. ■

For example, let G be the pendant graph in figure 1. Note that W ≠ φ. What is the upper bound for DOM(G) here? It is: (i) 1117.88 (by [5]); (ii) 143.95 (by [6]); (iii) **34.37** by proposition 2.4. The best upper bound is, evidently, by proposition 2.4.

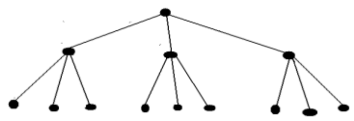

Fig. 1: A pendant graph G with n = 13, s = 3 and p = 9

The number $\delta = (2.7159^s)(1.7159^{-s-p})$ that improves the upper bound from 1.7159^n depends, of course, on s and p.

As another example, for a tree T with n = 6, s = p = 2, the upper bound on DOM(T) is: (i) 25.52 (by [5]); (ii) 9.92 (by [6]); (iii) 21.72 by proposition 2.4, and here [6] gives the best upper bound. In any case, proposition 2.4 improves the upper bound of [5] for the class of simple, loop-free connected pendant graphs.

3. The corresponding result for hypergraphs

Let H = (V, E) be a simple hypergraph. If x, y ∈ V are distinct, then x is *adjacent* to y in H if {x, y} ⊂ X for some X ∈ E. Let G be the graph on V defined as follows: If x, y ∈ V are distinct, then declare x and y to be adjacent in G if and only if x and y are adjacent in H. The graph G thus formed (on the vertices of H) is the *2-section* of H, and G is denoted by $(H)_2$.

3.1: Proposition: D is a minimal dominating set in H if and only if D is one in $(H)_2$.

Proof. Let D be a dominating set in H. If y ∈ V – D then y is adjacent to some x ∈ D in H, and so in $(H)_2$, in view of H and $(H)_2$ having the same vertex set V. Hence D is a dominating set in $(H)_2$. The converse is as straightforward.

From what is established in the preceding paragraph, the proposition follows at once. ■

3.2: Proposition: Let DOM(H) denote the number of minimal dominating sets in the hypergraph H = (V, E); and $n = |V|$. Then (i) DOM(H) $\leq 1.7159^n$, and (ii) if G =

$(H)_2$ is a pendant graph, then DOM(H) ≤ $(2.7159^s)(1.7159^{n-s-p})$, where s and p are as in 2.4 for G.

Proof. (i) is a direct consequence of 3.1, while (ii) is a result of 3.1 taken with either 2.3 or 2.4 when $(H)_2$ is a pendant graph. ■

4. Concluding remarks

Fomin et al [5] also gave a lower bound for the number of minimal dominating sets in a graph on n vertices, which is 1.5704^n. This applies to hypergraphs as well, in the light of 3.1. It was mentioned in [3] that 'there is a huge gap between the bounds 1.7159^n and 1.5704^n'. The improved upper bound $\delta(1.7159^n)$ (for pendant graphs, as in 2.3 and 2.4), reduces the gap between the upper and the lower bounds established in [5]. The more the number of pendant vertices in the graph, the narrower this gap is.

A future direction of research would be to study minimal dominating sets in other graph classes from the standpoint of an integer that has a considerably large frequency in the degree sequence of the vertices.

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